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RELATIONS BETWEEN ALGEBRAIC K-THEORY AND HERMITIAN K-THEORY

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This paper is a continuation of [4] where we computed the homology groups with coefficients of the infinite orthogonal and symplectic groups of an algebraically closed field F of characteristic $\neq 2$ and 0.

Since we have also proved in [4] that these homology groups depend only on the characteristic of F (if it is different from 2), in order to deal with the case of characteristic zero, it is sufficient to compute this homology when $F=\mathbb{C}$. More precisely, with the notations of [3] and [4], we shall prove the following statement:

Theorem 1. *The obvious topological group maps* $\mathrm{O}(\mathbb{C})\to\mathrm{O}(\mathbb{C})^{\mathrm{top}}$, $\mathrm{Sp}(\mathbb{C})\to\mathrm{Sp}(\mathbb{C})^{\mathrm{top}}$ *and* $\mathrm{U}(\mathbb{C})\to\mathrm{U}(\mathbb{C})^{\mathrm{top}}$ *induce isomorphisms*

$$H_i(B\mathrm{O}(\mathbb{C});\mathbb{Z}/q)\approx H_i(B\mathrm{O}(\mathbb{C})^{\mathrm{top}};\mathbb{Z}/q),$$

$$H_i(B\mathrm{Sp}(\mathbb{C});\mathbb{Z}/q)\approx H_i(B\mathrm{Sp}(\mathbb{C})^{\mathrm{top}};\mathbb{Z}/q),$$

$$H_i(B\mathrm{U}(\mathbb{C});\mathbb{Z}/q)\approx H_i(B\mathrm{U}(\mathbb{C})^{\mathrm{top}};\mathbb{Z}/q).$$

In this statement, $\mathrm{U}(\mathbb{C})$ is ${}_1\mathrm{O}(\mathbb{C})$ with $\mathbb{C}$ provided with the complex conjugation involution [3] and BG^{top} denotes in general the classifying space of the group G with its usual topology.

This theorem is proved below along the same lines as the theorem proved in [4], using essentially the fundamental theorem of Hermitian K-theory [3] and Gabber-Suslin's theorem [7]:

$$K_i(\mathbb{C};\mathbb{Z}/q)\approx K_i^{\mathrm{top}}(\mathbb{C};\mathbb{Z}/q)$$

In fact, since we have also $K_i(\mathbb{R};\mathbb{Z}/q)\approx K_i^{\mathrm{top}}(\mathbb{R};\mathbb{Z}/q)$ according to [7], we can prove in an analogous way the following theorem:

Theorem 2. *The topological group maps* $\mathrm{O}(\mathbb{R})\to\mathrm{O}(\mathbb{R})^{\mathrm{top}}$ *and* $\mathrm{Sp}(\mathbb{R})\to\mathrm{Sp}(\mathbb{R})^{\mathrm{top}}$ *induce isomorphisms*

$$H_i(BO(\mathbb{R}); \mathbb{Z}/q) \approx H_i(BO(\mathbb{R})^{\text{top}}; \mathbb{Z}/q),$$

$$H_i(BSp(\mathbb{R}); \mathbb{Z}/q) \approx H_i(BSp(\mathbb{R})^{\text{top}}; \mathbb{Z}/q).$$

If we put together Theorems 1 and 2 and some results of K. Vogtmann on the stability of the homology of these classical groups [8], [9], we get the following interesting corollary:

Corollary. *Let $F = \mathbb{R}$ or $\mathbb{C}$. Then we have the following homology isomorphisms:*

$$H_i(BO_{2n}(F); \mathbb{Z}/q) \approx H_i(BO_{2n}(F)^{\text{top}}; \mathbb{Z}/q) \quad \text{for } n \geq 3i,$$

$$H_i(BSp_{2n}(F); \mathbb{Z}/q) \approx H_i(BSp_{2n}(F)^{\text{top}}; \mathbb{Z}/q) \quad \text{for } n \geq 3i+3.$$

The theorems and the corollary give more evidence for the Friedlander–Milnor conjecture which is still open:

$$H_i(BG; \mathbb{Z}/q) \approx H_i(BG^{\text{top}}; \mathbb{Z}/q) \quad \text{for any Lie group } G.$$

Now the Theorems 1 and 2 are consequences of a theorem of a more general nature which we shall use elsewhere [5]:

Theorem 3. *Let A be a Banach algebra with involution. Then*

$$_\varepsilon W_1(A) \simeq {_\varepsilon W_1^{\text{top}}}(A), \qquad K_1(A; \mathbb{Z}/q) \approx K_1^{\text{top}}(A; \mathbb{Z}/q)$$

and

$$_\varepsilon L_1(A; \mathbb{Z}/q) \approx {_\varepsilon L_1^{\text{top}}}(A; \mathbb{Z}/q).$$

Let us assume moreover that $K_i(A; \mathbb{Z}/q) \approx K_i^{\text{top}}(A; \mathbb{Z}/q)$ for $2 \leq i \leq N$; then

$$_\varepsilon L_i(A; \mathbb{Z}/q) \approx {_\varepsilon L_i^{\text{top}}}(A; \mathbb{Z}/q) \quad \text{for } 2 \leq i \leq N.$$

Proof. For simplicity's sake, let us drop the letter A in the notations: we shall write K_i for $K_i(A)$, K_i^{top} for $K_i^{\text{top}}(A)$, etc. Following [4] we shall also write $\bar{K}_i, {_\varepsilon \bar{L}_i}, \ldots$ for the groups $K_i(A; \mathbb{Z}/q)$, $_\varepsilon L_i(A; \mathbb{Z}/q), \ldots$

It is clear that the maps $_\varepsilon L_1 \to {_\varepsilon L_1^{\text{top}}}$ and $_\varepsilon W_1 \xrightarrow{\gamma} {_\varepsilon W_1^{\text{top}}}$ are surjective. Let $\alpha \in {_\varepsilon W_1}$ be such that $\gamma(\alpha) = 0$. The argument used in Milnor's book [6, p. 58] shows that α is represented by a product of hyperbolic matrices in $_\varepsilon O_{n,n}(A)$ of the form

$$\alpha_i = \begin{pmatrix} 1 + a_i & b_i \\ c_i & 1 + d_i \end{pmatrix}$$

where a_i, b_i, c_i, d_i are $n \times n$ matrices close to 0 (for the Banach algebra topology). Now the argument used in [2, p. 405] shows that α_i is a product of hyperbolic matrices and ε-elementary matrices. Hence $\alpha = 0$ and $_\varepsilon W_1 \approx {_\varepsilon W_1^{\text{top}}}$.

We have two exact sequences

$$
\begin{array}{ccccccccc}
{}_{\varepsilon}L_1 & \xrightarrow{\times q} & {}_{\varepsilon}L_1 & \longrightarrow & {}_{\varepsilon}\bar{L}_1 & \longrightarrow & {}_{\varepsilon}L_0 & \xrightarrow{\times q} & {}_{\varepsilon}L_0 \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \downarrow & & \downarrow \\
{}_{\varepsilon}L_1^{\mathrm{top}} & \xrightarrow{\times q} & {}_{\varepsilon}L_1^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}\bar{L}_1^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}L_0 & \xrightarrow{\times q} & {}_{\varepsilon}L_0
\end{array}
$$

Since the kernel of σ is generated by matrices of the form $1+\beta$ where β is close to 0, this kernel is q-divisible (consider $\sqrt[q]{1+\beta}$). Therefore by diagram chasing, ${}_{\varepsilon}\bar{L}_1 \approx {}_{\varepsilon}\bar{L}_1^{\mathrm{top}}$. The same argument shows that $\bar{K}_1 \approx \bar{K}_1^{\mathrm{top}}$.

With the notations of [3] and [4], simple diagram chasing in the diagrams

$$
\begin{array}{ccccccccc}
K_1 & \longrightarrow & {}_{\varepsilon}L_1 & \longrightarrow & {}_{\varepsilon}U_0 & \longrightarrow & K_0 & \longrightarrow & {}_{\varepsilon}L_0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \| \\
K_1^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}L_1^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}U_0^{\mathrm{top}} & \longrightarrow & K_0 & \longrightarrow & {}_{\varepsilon}L_0
\end{array}
$$

$$
\begin{array}{ccccccccc}
{}_{\varepsilon}L_1 & \longrightarrow & K_1 & \longrightarrow & {}_{\varepsilon}V_0 & \longrightarrow & {}_{\varepsilon}L_0 & \longrightarrow & K_0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \| \\
{}_{\varepsilon}L_1^{\mathrm{top}} & \longrightarrow & K_1^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}V_0^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}L_0 & \longrightarrow & K_0
\end{array}
$$

shows that ${}_{\varepsilon}U_0 \approx {}_{\varepsilon}U_0^{\mathrm{top}}$ and that the map ${}_{\varepsilon}V_0 \to {}_{\varepsilon}V_0^{\mathrm{top}}$ is surjective with q-divisible kernel. Let us consider the diagram of exact sequences

$$
\begin{array}{ccccccccc}
{}_{\varepsilon}U_1 & \xrightarrow{\times q} & {}_{\varepsilon}U_1 & \longrightarrow & {}_{\varepsilon}\bar{U}_1 & \longrightarrow & {}_{\varepsilon}U_0 & \xrightarrow{\times q} & {}_{\varepsilon}U_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} \\
{}_{\varepsilon}U_1^{\mathrm{top}} & \xrightarrow{\times q} & {}_{\varepsilon}U_1^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}\bar{U}_1^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}U_0^{\mathrm{top}} & \xrightarrow{\times q} & {}_{\varepsilon}U_0^{\mathrm{top}}
\end{array}
$$

According to the fundamental theorem of Hermitian K-theory ([2] and [3]), ${}_{\varepsilon}U_1 \approx {}_{-\varepsilon}V_0$ and ${}_{\varepsilon}U_1^{\mathrm{top}} \approx {}_{-\varepsilon}V_0^{\mathrm{top}}$. Therefore, the map ${}_{\varepsilon}U_1 \to {}_{\varepsilon}U_1^{\mathrm{top}}$ is surjective with q-divisible kernel and the map ${}_{\varepsilon}\bar{U}_1 \to {}_{\varepsilon}\bar{U}_1^{\mathrm{top}}$ is an isomorphism. Using the fundamental theorem of Hermitian K-theory again (the mod q version), we also have an isomorphism ${}_{\varepsilon}\bar{V}_0 \approx {}_{\varepsilon}\bar{V}_0^{\mathrm{top}}$ for any ε.

Finally, if $\bar{K}_2 \approx \bar{K}_2^{\mathrm{top}}$, the diagram of exact sequences

$$
\begin{array}{ccccccccc}
{}_{\varepsilon}\bar{L}_2 & \longrightarrow & \bar{K}_2 & \longrightarrow & {}_{\varepsilon}\bar{V}_1 & \longrightarrow & {}_{\varepsilon}\bar{L}_1 & \longrightarrow & \bar{K}_1 \\
\downarrow & & {\scriptstyle \approx}\downarrow & & \downarrow & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} \\
{}_{\varepsilon}\bar{L}_2^{\mathrm{top}} & \longrightarrow & \bar{K}_2^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}\bar{V}_1^{\mathrm{top}} & \longrightarrow & {}_{\varepsilon}\bar{L}_1^{\mathrm{top}} & \longrightarrow & \bar{K}_1^{\mathrm{top}}
\end{array}
$$

implies the surjectivity of the map ${}_\varepsilon\bar{V}_1 \to {}_\varepsilon\bar{V}_1^{\mathrm{top}}$.

For a fixed A and any ε, let us call (H_i) and (H'_i) the following hypotheses:

$$(\mathrm{H}_i)\quad \left\{\begin{array}{ll} {}_\varepsilon\bar{L}_i(A)\to{}_\varepsilon\bar{L}_i^{\mathrm{top}}(A) & \text{is an isomorphism} \\ {}_\varepsilon\bar{V}_{i-1}(A)\to{}_\varepsilon\bar{V}_{i-1}^{\mathrm{top}}(A) & \text{is an isomorphism} \\ {}_\varepsilon\bar{V}_i(A)\to{}_\varepsilon\bar{V}_i^{\mathrm{top}}(A) & \text{is an epimorphism.} \end{array}\right. \quad (\mathrm{H}'_i)$$

According to what we have just proved, (H_1) is satisfied if $N\geq 2$. Therefore, it is sufficient to prove that $(\mathrm{H}_i)\Rightarrow(\mathrm{H}_{i+1})$ if $1\leq i<N-1$ and that $(\mathrm{H}_i)\Rightarrow(\mathrm{H}'_{i+1})$ if $1\leq i\leq N-1$.

Since ${}_\varepsilon\bar{U}_j\approx{}_{-\varepsilon}\bar{V}_{j-1}$, and ${}_\varepsilon\bar{U}_j{}^{\mathrm{top}}\approx{}_{-\varepsilon}\bar{V}_{j-1}^{\mathrm{top}}$, we have the following commutative diagram with three isomorphisms and one epimorphism

$$\begin{array}{ccccccccc} {}_\varepsilon\bar{U}_{i+1} & \longrightarrow & \bar{K}_{i+1} & \longrightarrow & {}_\varepsilon\bar{L}_{i+1} & \longrightarrow & {}_\varepsilon\bar{U}_i & \longrightarrow & \bar{K}_i \\ \downarrow & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \downarrow\approx \\ {}_\varepsilon\bar{U}_{i+1}^{\mathrm{top}} & \longrightarrow & \bar{K}_{i+1}^{\mathrm{top}} & \longrightarrow & {}_\varepsilon\bar{L}_{i+1}^{\mathrm{top}} & \longrightarrow & {}_\varepsilon\bar{U}_i^{\mathrm{top}} & \longrightarrow & \bar{K}_i^{\mathrm{top}} \end{array}$$

if $1\leq i\leq N-1$. Therefore ${}_\varepsilon\bar{L}_{i+1}\approx{}_\varepsilon\bar{L}_{i+1}^{\mathrm{top}}$ by the Five Lemma.

In the same way, assuming again (H_i) for $1\leq i<N-1$, the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} {}_\varepsilon\bar{L}_{i+1} & \longrightarrow & \bar{K}_{i+1} & \longrightarrow & {}_\varepsilon\bar{V}_i & \longrightarrow & {}_\varepsilon\bar{L}_i & \longrightarrow & \bar{K}_i \\ \downarrow & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \downarrow\approx \\ {}_\varepsilon\bar{L}_{i+1}^{\mathrm{top}} & \longrightarrow & \bar{K}_{i+1}^{\mathrm{top}} & \longrightarrow & {}_\varepsilon\bar{V}_{i+1}^{\mathrm{top}} & \longrightarrow & {}_\varepsilon\bar{L}_{i+1}^{\mathrm{top}} & \longrightarrow & \bar{K}_{i+1} \end{array}$$

implies ${}_\varepsilon\bar{V}_i\approx{}_\varepsilon\bar{V}_i^{\mathrm{top}}$. Finally, if $1\leq i<N-1$, the diagram of exact sequences

$$\begin{array}{ccccccccc} {}_\varepsilon\bar{L}_{i+2} & \longrightarrow & \bar{K}_{i+2} & \longrightarrow & {}_\varepsilon V_{i+1} & \longrightarrow & {}_\varepsilon\bar{L}_{i+1} & \longrightarrow & \bar{K}_{i+1} \\ \downarrow & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \downarrow\approx \\ {}_\varepsilon\bar{L}_{i+2}^{\mathrm{top}} & \longrightarrow & \bar{K}_{i+2}^{\mathrm{top}} & \longrightarrow & {}_\varepsilon\bar{V}_{i+1}^{\mathrm{top}} & \longrightarrow & {}_\varepsilon\bar{L}_{i+1}^{\mathrm{top}} & \longrightarrow & \bar{K}_{i+1}^{\mathrm{top}} \end{array}$$

implies the surjectivity of the map ${}_\varepsilon\bar{V}_{i+1}\to{}_\varepsilon\bar{V}_{i+1}^{\mathrm{top}}$.

This proof of Theorem 3 is an illustration of a principle which roughly states that a general theorem for $K_i(A;\mathbb{Z}/q)$ implies a general theorem for ${}_\varepsilon L_i(A;\mathbb{Z}/q)$. As an other example we offer the following theorem:

Theorem 4. *Let $A\to B$ be a map of rings with involution such that*

$$K_0(A)\to K_0(B), \qquad {}_\varepsilon L_0(A)\to{}_\varepsilon L_0(B), \qquad {}_\varepsilon W_1(A)\to{}_\varepsilon W_1(B)$$

are isomorphisms, $K_1A)\to K_1(B)$ *and* ${}_\varepsilon L_1(A)\to{}_\varepsilon L_1(B)$ *are surjective with q-divisible kernels (for all* ε*). Then*

$${}_\varepsilon L_1(A\colon \mathbb{Z}/q)\approx{}_\varepsilon L_1(B;\mathbb{Z}/q) \quad \textit{and} \quad K_1(A;\mathbb{Z}/q)\approx K_1(B;\mathbb{Z}/q).$$

Let us assume moreover that $K_i(A;\mathbb{Z}/q)\approx K_i(B;\mathbb{Z}/q)$ *for all i such that* $2\le i\le N$ *and if q is even that there exists an element* λ *of A such that* $\lambda+\bar\lambda=1$*. Then*

$${}_\varepsilon L_i(A;\mathbb{Z}/q)\approx{}_\varepsilon L_i(B;\mathbb{Z}/q) \quad \textit{and} \quad \mathrm{H}_i({}_\varepsilon\mathrm{O}(A);\mathbb{Z}/q)\approx\mathrm{H}_i({}_\varepsilon\mathrm{O}(B);\mathbb{Z}/q)$$

for any ε *and* $1\le i\le N$*.*

The proof of this theorem is completely analogous to that of Theorem 3: we only have to replace K_i^{top}, ${}_\varepsilon L_i^{\mathrm{top}},\ldots$, by $K_i(B)$, ${}_\varepsilon L_i(B)$. According to Gabber and Suslin ([1] and [7]), the hypotheses of the theorem are fulfilled when $B=A/I$ where (A,I) is a Henselian pair with $1/q\in A$ such that I is invariant by the involution and such that there exists $\lambda\in A$ with $\lambda+\bar\lambda=1$ (if q is even). For instance,

$${}_\varepsilon L_i(\hat{\mathbb{Z}}_p;\mathbb{Z}/p)\approx{}_\varepsilon L_i(\mathbb{Z}/p;\mathbb{Z}/q) \quad \text{with } p\neq 2 \quad \text{if } q \text{ is even.}$$

Note. Very recently, J.F. Jardine [10] has proved analogous results using also [2], [3] and [7]. In particular, he has proved Theorem 1 for O(ℂ) and Sp(ℂ) and a corollary of Theorem 4 for a Henselian pair (A,I) where $B=A/I$ is a field.

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